On the exact constant in Jackson-Stechkin inequality for the uniform metric

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Abstract

The classical Jackson-Stechkin inequality estimates the value of the best uniform approximation of a periodic function f by trigonometric polynomials of degree $\le n-1$ in terms of its r-th modulus of smoothness $\omega_r(f,\delta)$. It reads

$$E_{n-1}(f) \le c_r\, \omega_r\Big(f, \frac{2\pi}{n}\Big),$$

where c_r is *some* constant that depends only on r. It was known that c_r admits the estimate $c_r < r^{ar}$ and, basically, nothing else could be said about it.

The main result of this paper is in establishing that

$$(1 - \tfrac{1}{r+1})\, \gamma_r^* \le c_r < 5\, \gamma_r^*, \qquad \gamma_r^* = \frac{1}{\binom{r}{\lfloor \frac{r}{2} \rfloor}} \asymp \frac{r^{1/2}}{2^r},$$

i.e., that the Stechkin constant c_r, far from increasing with r, does in fact decay exponentially fast. We also show that the same upper bound is valid for the constant $c_{r,p}$ in the Stechkin inequality for L_p-metrics with $p \in [1,\infty)$, and for small r we present upper estimates which are sufficiently close to $1 \cdot \gamma_r^*$.

1 Introduction

The classical Jackson-Stechkin inequality estimates the value of the best uniform approximation of a periodic function f by trigonometric polynomials of degree $\le n-1$ in terms of its r-th modulus of smoothness $\omega_r(f,\delta)$. It reads

$$E_{n-1}(f) \le c_r\, \omega_r\Big(f, \frac{2\pi}{n}\Big), \tag{1.1}$$

where c_r is *some* constant which depends only on r (see [10] or [3, p.205]).

Besides the case $r = 1$, hardly any attempts have been made to find the best value of this constant c_r, or even to determine its dependence on r. Stechkin's original proof [10] (as well as alternative ones) allows to obtain the estimate $c_r < r^{ar}$, and, basically, nothing else could be said about it.

The main result of this paper is in establishing that

$$(1 - \tfrac{1}{r+1})\, \gamma_r^* \le c_r < 5\, \gamma_r^*, \qquad \gamma_r^* = \frac{1}{\binom{r}{\lfloor \frac{r}{2} \rfloor}} \asymp \frac{r^{1/2}}{2^r}, \tag{1.2}$$

[0] *AMS classification*: Primary 41A17, 41A44, 42A10.

[0] *Key words and phrases*: Jackson-Stechkin inequality, r-th modulus of smoothness, exact constants.

i.e., that the Stechkin constant c_r, far from increasing with r, does in fact decay exponentially fast.

We also show that the same upper bound is valid for the constant $c_{r,p}$ in the Stechkin inequality for L_p-metrics with $p \in [1,\infty)$, and for small r we present upper estimates which are sufficiently close to $1 \cdot \gamma_r^*$.

In retrospect, such a result could have been anticipated, since for trigonometric approximation in L_2-metric, already in 1967, Chernykh [2] established that

$$E_{n-1}(f)_2 \le c_{r,2}\, \omega_r\Big(f, \frac{2\pi}{n}\Big)_2, \qquad c_{r,2} = \frac{1}{\sqrt{\binom{2r}{r}}} \asymp \frac{r^{1/4}}{2^r}\,, \tag{1.3}$$

proving also that such a $c_{r,2}$ is best possible (for the argument $\delta = \frac{2\pi}{n}$ in $\omega_r(f,\delta)$). However, this result was based on Fourier technique for L_2-approximation and that does not work in other L_p-metrics.

Our method of proving (1.2) is based on deriving first the intermediate inequality

$$\|f\| \le c_{n,r}(\delta)\, \omega_r(f,\delta)\,, \qquad f \in T_{n-1}^{\perp}, \tag{1.4}$$

which is valid for the functions f which are orthogonal to the trigonometric polynomials of degree $\le n-1$. This may be viewed as a difference analogue of the classical Bohr-Favard inequality for differentiable functions

$$\|f\| \le \frac{F_r}{n^r}\, \|f^{(r)}\|\,, \qquad f \in T_{n-1}^{\perp},$$

and is of independent interest.

We make a pass from the Bohr-Favard-type inequality (1.4) to the Stechkin one (1.1) by approximating f with the de la Vallée Poussin sums $v_{m,n}(f)$ and using the fact that

$$f - v_{m,n}(f) \in T_m^{\perp}\,, \quad \omega_r(f - v_{m,n}(f),\delta) \le (1 + \|v_{m,n}\|)\, \omega_r(f,\delta).$$

With that we arrive at the inequality

$$E_{n-1}(f) \le \|f - v_{m,n}(f)\| \le c_{m,n,r}(\delta)\omega_r(f,\delta)\,,$$

where we finally minimize the resulting constant over m, for given r, n and δ.

2 Results

For a continuous 2π-periodic function f, we denote by $E_{n-1}(f)$ the value of best approximation of f by trigonometric polynomials of degree $\le n-1$ in the uniform norm,

$$E_{n-1}(f) := \inf_{\tau \in T_{n-1}} \|f - \tau\|\,,$$

and by $\omega_r(f,\delta)$ its r-th modulus of smoothness with the step δ,

$$\omega_r(f,\delta) := \sup_{0<h\le\delta} \|\Delta_h^r(f,\cdot)\|, \qquad \Delta_h^r(f,x) = \sum_{i=0}^{r} (-1)^i \binom{r}{i} f(x+ih)\,,$$

where $\Delta_h^r(f,x)$ is the forward difference of order r of f at x with the step h.

We will study the best constant $K_{n,r}(\delta)$ in the Stechkin inequality

$$E_{n-1}(f) \le K_{n,r}(\delta)\,\omega_r(f,\delta)\,,$$

i.e., the quantity

$$K_{n,r}(\delta) := \sup_{f\in C} \frac{E_{n-1}(f)}{\omega_r(f,\delta)}\,,$$

which depends on the given parameters $n, r \in \mathbb{N}$ and $\delta \in [0, 2\pi]$.

In such a setting (which goes back to Korneichuk and Chernykh) we may safely consider $\delta = \frac{\alpha\pi}{n}$ with some α not necessarily 1 or 2. The choice of particular δ's can be motivated by two reasons:

1) "nice" look and/or tradition: $\delta = \frac{\pi}{n}$, or $\delta = \frac{2\pi}{n}$, or (why not) $\delta = \frac{1}{n}$, and alike;

2) "nice" result:

$$\sup_{f\in C} \frac{E_{n-1}(f)}{\omega_r(f,\delta)} \asymp c_{n,r}(\delta)\,.$$

Ideally, both approaches should be combined to provide nice results for nice δ's, but that happens not very often.

In this paper we obtain the following results.

1) First of all, we show that the exact order of the Stechkin constant $K_{n,r}(\delta)$ at $\delta = \frac{2\pi}{n}$ (and in fact at any $\delta \in [\frac{2\pi}{n}, \frac{\pi}{r}]$) is $r^{1/2}2^{-r}$, namely

$$K_{n,r}(\tfrac{2\pi}{n}) \asymp \gamma_r^* \asymp \frac{r^{1/2}}{2^r}\,,$$

where

$$\gamma_r^* = \frac{1}{\binom{r}{\lfloor \frac{r}{2}\rfloor}} = \begin{cases} \frac{1}{\binom{2k}{k}}, & r = 2k; \\ \frac{1}{\binom{2k-1}{k-1}}, & r = 2k-1. \end{cases}$$

Moreover, we locate the exact value of this constant within quite a narrow interval.

Theorem 1. *We have*

$$c_r'(\tfrac{2\pi}{n})\,\gamma_r^* \;\le\; \sup_{f\in C} \frac{E_{n-1}(f)}{\omega_r(f,\frac{2\pi}{n})} \;\le\; c_r(\tfrac{2\pi}{n})\,\gamma_r^*\,,$$

where

$$c_r'(\tfrac{2\pi}{n}) = \begin{cases} 1 - \frac{1}{r+1}, & r = 2k-1; \\ 1, & r = 2k; \end{cases} \qquad n \ge 2r,$$

and

$$c_r(\tfrac{2\pi}{n}) = 5, \qquad n \ge 1.$$

Surprising is the fact that, in this theorem, the upper estimate is provided by one and the same linear method of approximation that works for all r simultaneously. Namely, for any r, the de la Vallée Poussin operator $v_{m,n}$ with $m = \lfloor \frac{8}{9} n \rfloor$ provides

$$\|f - v_{m,n}(f)\| \le 5\,\gamma_r^* \omega_r\Big(f, \frac{2\pi}{n}\Big)\,, \qquad \forall r \in \mathbb{N}.$$

2) Next, we show that the value of the constant $c_r(\delta)$ remains bounded uniformly in r and n also for $\frac{\pi}{n} < \delta < \frac{2\pi}{n}$ (but it grows to infinity as δ approaches $\frac{\pi}{n}$).

Theorem 2. *For any $\alpha > 1$, there exists a constant c_α which depends only on α such that*

$$E_{n-1}(f) \le c_\alpha\, \gamma_r^*\, \omega_r\Big(f, \frac{\alpha\pi}{n}\Big), \qquad n \ge 1\,.$$

3) Thirdly, although we did not succeed to reach the argument $\delta = \frac{\pi}{n}$ with an absolute constant in front of $\gamma_r^*\, \omega_r(f,\delta)$, we prove that this constant grows like $\mathcal{O}(\sqrt{r}\ln r)$ at most.

Theorem 3. *For $\delta = \frac{\pi}{n}$, we have the estimate*

$$E_{n-1}(f) \le c_r(\tfrac{\pi}{n})\, \gamma_r^*\, \omega_r\Big(f, \frac{\pi}{n}\Big), \qquad c_r(\tfrac{\pi}{n}) = \mathcal{O}(\sqrt{r}\ln r)\,, \qquad n \ge 1\,.$$

4) Fourthly, for small r, the general upper bound $c_r(\frac{2\pi}{n}) = 5$ can be decreased to the values that are quite close to the lower bound $c_r' \approx 1$, thus giving support to the (upcoming) conjecture that $K_{n,r}(\delta) \le 1 \cdot \gamma_r^*$ for $\delta \ge \frac{\pi}{n}$.

Theorem 4. *For $\delta = \frac{\pi}{n}$ and $\delta = \frac{2\pi}{n}$, we have*

$$E_{n-1}(f) \le c_r(\delta)\, \gamma_r^*\, \omega_r(f,\delta)\,,$$

where $c_{2k-1}(\delta) = c_{2k}(\delta)$, and the values of $c_{2k}(\delta)$ are given below

$c_2(\frac{\pi}{n})$	$c_4(\frac{\pi}{n})$
$1\frac{1}{4}$	$2\frac{7}{10}$

,

$c_2(\frac{2\pi}{n})$	$c_4(\frac{2\pi}{n})$	$c_6(\frac{2\pi}{n})$
$1\frac{1}{16}$	$1\frac{1}{9}$	$1\frac{1}{2}$

.

5) Finally, all upper estimates in Theorems 1-4 remain valid for any $p \in [1,\infty]$. (There is no need to give a separate proof of this statement, since all the inequalities we used in the text still hold for the L_p-metrics, $1 \le p < \infty$, in particular the Bohr-Favard inequality (4.1) and the inequalities of §5 involving the norms of the de la Vallée Poussin operator.)

Theorem 5. *For any $p \in [1,\infty]$, we have*

$$E_{n-1}(f)_p \le c_r(\delta)\, \gamma_r^*\, \omega_r(f,\delta)_p\,,$$

with the same constants $c_r(\delta)$ and the same δ's as in Theorems 1–4. In particular,

$$E_{n-1}(f)_p \le 5\, \gamma_r^*\, \omega_r\Big(f, \frac{2\pi}{n}\Big)_p\,.$$

The latter L_p-estimate is hardly of the right order for $1 < p < \infty$ because $\gamma_r^* \asymp r^{1/2}\, 2^{-r}$, while, for $p = 2$, Chernykh's result (1.3) says that $K_{n,r}(\frac{2\pi}{n})_2 \asymp r^{1/4}\, 2^{-r}$, so one may guess that

$$K_{n,r}(\tfrac{2\pi}{n})_p \asymp r^{\max(1/2p,1/2p')}\, 2^{-r}\,.$$

This guess is partially based on the results of Ivanov [7] who obtained such an upper bound for the values $K_{n,r}(\delta)_p$ with relatively large $\delta = \frac{\pi r^{1/3}}{n}$, and proved that, for $p \in [2,\infty]$, the order of the lower bounds is the same.

6) The value $\delta = \frac{\pi}{n}$ is critical in the sense that the Stechkin constant $K_{n,r}(\delta)$ and the constant γ_r^* are no longer of the same (exponential) order for $\delta = \frac{\alpha\pi}{n}$ with $\alpha < 1$. Indeed, in this case, with $f_0(x) := \cos nx$, we have $\omega_r(f_0, \delta) = 2^r \sin^r \frac{\alpha\pi}{2}$ (see (7.2)) and $E_{n-1}(f_0) = 1$, so that, for $\alpha < 1$, we have

$$\frac{K_{n,r}(\frac{\alpha\pi}{n})}{\gamma_r^*} > \frac{cr^{1/2}}{\sin^r \frac{\alpha\pi}{2}} > c_\alpha \lambda_\alpha^r, \qquad \lambda_\alpha > 1\,.$$

This being said, a natural question arises from the two estimates

$$K_{n,r}(\tfrac{2\pi}{n}) \asymp \gamma_r^*, \qquad K_{n,r}(\tfrac{\pi}{n}) \le c\sqrt{r}\ln r \cdot \gamma_r^*$$

whether an extra factor at $\delta = \frac{\pi}{n}$ is essential. We believe it is not, and we are making the following brave conjecture.

Conjecture 2.1 *For all $r \in \mathbb{N}$, we have*

$$\sup_{n\in\mathbb{N}} K_{n,r}(\tfrac{\pi}{n}) := \sup_{n\in\mathbb{N}} \sup_{f\in C} \frac{E_{n-1}(f)}{\omega_r(f, \frac{\pi}{n})} = 1 \cdot \gamma_r^*, \qquad \gamma_r^* = \frac{1}{\binom{r}{\lfloor \frac{r}{2} \rfloor}}\,.$$

(Our point is mainly about the upper bound, namely that $K_{n,r}(\delta) \le 1 \cdot \gamma_r^*$, for any $\delta \ge \frac{\pi}{n}$. The lower bound for even $r = 2k$ is established in this paper, while for odd r we guess that $K_{n,r}(\delta)$ tends to γ_r^* at $\delta = \frac{\pi}{n}$ for large n, but for $\delta > \frac{\pi}{n}$ it takes smaller values.)

This conjecture is true for $r = 1$, for in this case we have Korneichuk's result [9]:

$$1 - \frac{1}{2n} \le K_{n,1}(\tfrac{\pi}{n}) < 1.$$

For $r = 2$, the conjecture gives the estimate $K_{n,2}(\frac{\pi}{n}) = \frac{1}{2}$ which is (to a certain extent) stronger than Korneichuk's one (because $\omega_2(f,\delta) \le 2\omega_1(f,\delta)$), so it would be interesting to prove (or to disprove) it in this particular case. Meanwhile, acccording to Theorems 1 and 4, we have

$$\frac{1}{2} \le K_{n,2}(\tfrac{\pi}{n}) \le \frac{5}{8}\,, \qquad \frac{1}{2} \le K_{n,2}(\tfrac{2\pi}{n}) \le \frac{17}{32}\,.$$

For arbitrary r, it seems unlikely that the value of the Stechkin constant will ever be precisely determined, but it would be a good achievement to narrow the interval for $K_{n,r}(\frac{2\pi}{n})$, say, to $[\gamma_r^*, 2\gamma_r^*]$, and to settle down the correct order of $K_{n,r}(\frac{\pi}{n})$ with respect to r.

7) We finish this section with the remark that if, with some constant $c(\delta)$, the inequality

$$E_{n-1}(f) \le c(\delta)\, \gamma_r^*\, \omega_r(f,\delta)$$

is true for an even $r = 2k$, then it is true for the odd $r = 2k-1$ too, with the same constant $c(\delta)$. Indeed, since $\gamma_{2k-1}^* = 2\gamma_{2k}^*$, and $\omega_{2k}(f,\delta) \le 2\,\omega_{2k-1}(f,\delta)$, we have

$$\begin{aligned} E_{n-1}(f) &\le c(\delta)\,\gamma_{2k}^*\,\omega_{2k}(f,\delta) \\ &\le c(\delta)\,\gamma_{2k}^* \cdot 2\,\omega_{2k-1}(f,\delta) = c(\delta)\,\gamma_{2k-1}^*\,\omega_{2k-1}(f,\delta)\,. \end{aligned}$$

Therefore, it is sufficient to prove upper estimates only for even $r = 2k$.

3 Smoothing operators

Here, we present the general idea of our method.

1) For a fixed k, with

$$\widehat{\Delta}_t^{2k}(f,x) := \sum_{i=-k}^{k} (-1)^i \binom{2k}{k+i} f(x+it)$$

being the central difference of order $2k$ with the step t, and with ϕ_h being an integrable function which satisfies conditions

$$a) \quad \phi_h(t) = \phi_h(-t), \qquad b) \quad \operatorname{supp} \phi_h = [-h,h], \qquad c) \quad \int_{\mathbb{R}} \phi_h(t)\,dt = 1, \tag{3.1}$$

consider the following operator

$$W_h(f,x) := \frac{1}{\binom{2k}{k}} \int_{\mathbb{R}} \widehat{\Delta}_t^{2k}(f,x)\phi_h(t)\,dt\,. \tag{3.2}$$

If a given subspace $\mathcal{S}$ is invariant under the operator W_h, and if the restriction W_h to $\mathcal{S}$ has a bounded inverse, then, for any $f \in \mathcal{S}$, we have a trivial estimate

$$\|f\| \le \|W_h^{-1}\|_{\mathcal{S}} \|W_h(f)\|\,, \qquad f \in \mathcal{S}. \tag{3.3}$$

It follows immediately from the definition that

$$\|W_h(f)\| \le \|\phi_h\|_1\, \gamma_{2k}^*\, \omega_{2k}(f,h), \qquad \gamma_{2k}^* = \frac{1}{\binom{2k}{k}}\,, \tag{3.4}$$

and we arrive at the following inequality:

$$\|f\| \le c_{2k}(h)\, \gamma_{2k}^*\, \omega_{2k}(f,h)\,, \qquad c_{2k}(h) = \|\phi_h\|_1 \|W_h^{-1}\|_{\mathcal{S}}\,, \tag{3.5}$$

valid for all functions f from a given subspace $\mathcal{S}$.

2) Next, we present W_h as $W_h = I - U_h$ what allows us to get some bounds for $\|W_h^{-1}\|$ in (3.5) in terms of U_h.

To this end, for integer i (and, in fact, for any i), define the dilations ϕ_{ih} and the convolution operators I_{ih} by the rule

$$\phi_{ih}(t) := \tfrac{1}{i}\phi_h(\tfrac{t}{i})\,, \qquad I_{ih}(f) := f * \phi_{ih} := \int_{\mathbb{R}} f(\cdot - t)\phi_{ih}(t)\,dt\,.$$

Then, taking into account that

$$\int_{\mathbb{R}} f(x-it)\phi_h(t)\,dt = \int_{\mathbb{R}} f(x-\tau)\tfrac{1}{i}\phi_h(\tfrac{\tau}{i})\,d\tau = I_{ih}(f)\,,$$

and that also $I_{ih} = I_{-ih}$ (because ϕ_{ih} is even), we may put W_h in the following form:

$$W_h = \frac{1}{\binom{2k}{k}} \sum_{i=-k}^{k} (-1)^i \binom{2k}{k+i} I_{ih} = I - 2\sum_{i=1}^{k} (-1)^{i+1} a_i I_{ih}, \qquad a_i := \frac{\binom{2k}{k+i}}{\binom{2k}{k}}\,.$$

So, with the further notations

$$U_h := 2\sum_{i=1}^{k}(-1)^{i+1}a_i I_{ih}\,, \qquad \psi_{kh} := 2\sum_{i=1}^{k}(-1)^{i+1}a_i\phi_{ih}\,,$$

we obtain

$$W_h = I - U_h\,, \qquad U_h(f) = f * \psi_{kh}\,.$$

Respectively, we may rewrite the inequality (3.5) in the following way.

Lemma 3.1 *If the opeartor $(I-U_h)^{-1}$ is bounded on a given subspace $\mathcal{S}$, then, for all $f \in \mathcal{S}$, we have*

$$\|f\| \le c_{2k}(h)\,\gamma_{2k}^*\,\omega_{2k}(f,h)\,, \qquad c_{2k}(h) = \|\phi_h\|_1 \|(I-U_h)^{-1}\|_{\mathcal{S}}\,.$$

3) Now, we call upon elementary properties of Banach algebras (see, e.g., Kantorovich, Akilov [8, Chapter 5, § 4]) for the claim that if an operator $U : \mathcal{S} \to \mathcal{S}$ satisfies $\sum_{m=0}^{\infty}\|U^m\| < \infty$, then the operator $I - U$ is invertible, and the norm of its inverse admits the estimate

$$\|(I-U)^{-1}\|_{\mathcal{S}} \le \sum_{m=0}^{\infty}\|U^m\|_{\mathcal{S}}\,. \tag{3.6}$$

Proposition 3.2 *If ϕ_h is such that $\sum_{m=0}^{\infty}\|U_h^m\|_{\mathcal{S}} = A_h < \infty$, then, for any $f \in \mathcal{S}$, we have*

$$\|f\| \le c_{2k}(h)\,\gamma_{2k}^*\,\omega_{2k}(f,h), \qquad c_{2k}(h) = A_h\,\|\phi_h\|_1\,.$$

4) Finally, let us make a short remark about the structure of the subspaces $\mathcal{S}$ that may go into consideration. It is clear that, if $\mathcal{S}$ is shift-invariant, i.e., together with f it contains also $f(\cdot + t)$ for any t, then $\mathcal{S}$ is invariant under the action of W_h for any h. A typical example is a subspace $\mathcal{S}$ that contains (or does not contain) certain monomials $\binom{\cos kx}{\sin kx}$.

We will consider $\mathcal{S} = T_{n-1}^{\perp}$, the subspace of functions which are orthogonal to trigonometric polynomials of degree $\le n-1$.

4 A difference analogue of the Bohr-Favard inequality

Denote by $T_{n-1}^{\perp}$ the set of functions f which are orthogonal to T_{n-1}, i.e., such that

$$\int_{-\pi}^{\pi} f(x)\tau(x)\,dx = 0, \qquad \forall \tau \in T_{n-1}\,.$$

The Bohr-Favard inequality for such functions reads

$$\|f\| \le \frac{F_r}{n^r}\|f^{(r)}\|, \qquad f \in T_{n-1}^{\perp}\,, \tag{4.1}$$

where F_r are the Favard constants, which are usually defined by the formula

$$F_r := \frac{4}{\pi}\sum_{i=0}^{\infty}\frac{(-1)^{i(r+1)}}{(2i+1)^{r+1}}\,,$$

and which satisfy the following relations:

$$F_0 = 1 < F_2 = \frac{\pi^2}{8} < \cdots < \frac{4}{\pi} < \cdots < F_3 = \frac{\pi^3}{24} < F_1 = \frac{\pi}{2}\,.$$

In this section we obtain a difference analogue of the Bohr-Favard inequality in the form

$$\|f\| \le c_{n,2k}(h)\,\gamma_{2k}^*\,\omega_{2k}(f,h), \qquad f \in T_{n-1}^{\perp}\,,$$

using the approach from the previous section (Proposition 3.2). Namely, we consider the operator

$$U_h = 2\sum_{i=1}^{k}(-1)^{i+1} a_i I_{ih}, \qquad a_i = \binom{2k}{k+i}\Big/\binom{2k}{k}\,,$$

with the following specific choice of I_h (and respectively of ϕ_h):

$$I_h(f,x) := \frac{1}{h^2}\int_{-h/2}^{h/2}\int_{-h/2}^{h/2} f(x-t_1-t_2)\,dt_1\,dt_2\,,$$

i.e., taking $I_h(f)$ as the Steklov function of order 2. It is known that $I_h(f,x) = f * \phi_h$, where

$$\phi_h(t) = \begin{cases} \frac{1}{h}(1-\frac{|t|}{h}), & t \in [-h,h],\\ 0, & \text{otherwise,}\end{cases}$$

i.e., ϕ_h is the L_1-normalized B-spline of order 2 (the hat-function) with the step-size h supported on $[-h,h]$. We also have

$$I_{ih}''(f,x) = -\frac{1}{(ih)^2}\widehat{\Delta}_{ih}^2 f(x) = \frac{1}{(ih)^2}[f(x-ih) - 2f(x) + f(x+ih)]\,.$$

We denote by $\|U_h\|_{T_{n-1}^{\perp}}$ the norm of the operator U_h on the space $T_{n-1}^{\perp}$.

Lemma 4.1 *We have*

$$\|U_h''\| \le \frac{\pi^2\mu^2}{h^2}\,, \qquad \mu^2 := \mu_{2k}^2 := \frac{8}{\pi^2}\sum_{\text{odd}\, i}^{k}\frac{a_i}{i^2} < 1\,. \tag{4.2}$$

Proof. 1) We have

$$\begin{aligned} U_h''(f,x) &= 2\sum_{i=1}^{k}(-1)^{i+1}\,a_i I_{ih}''(f,x)\\ &= 2\sum_{i=1}^{k}(-1)^{i+1}\,\frac{a_i}{(ih)^2}\Big[f(x-ih) - 2f(x) + f(x+ih)\Big]\\ &= \frac{2}{h^2}\sum_{i=1}^{k}(-1)^{i+1}\,\frac{a_i}{i^2}\Big[f(x-ih) - 2f(x) + f(x+ih)\Big] \qquad (4.3)\\ &= \frac{2}{h^2}\sum_{i=1}^{k}(-1)^{i+1}a_i'\Big[-2f(x)\Big] + \frac{2}{h^2}\sum_{i=1}^{k}(-1)^{i+1}\,a_i'\Big[f(x-ih)+f(x+ih)\Big]\,,\end{aligned}$$

where in the last line we put $a'_i = \frac{a_i}{i^2}$. Hence,

$$\begin{aligned}\frac{h^2}{4}\,\frac{\|U''_h(f)\|}{\|f\|} &\le \Big|\sum_{i=1}^k(-1)^{i+1}a'_i\Big| + \sum_{i=1}^k |a'_i| \;=\; \sum_{i=1}^k(-1)^{i+1}a'_i + \sum_{i=1}^k a'_i \\ &= \;2\sum_{\text{odd}\, i}^k a'_i \;=\; 2\sum_{\text{odd}\, i}^k \frac{a_i}{i^2} =: \frac{\pi^2}{4}\,\mu^2\end{aligned}$$

i.e.,

$$\|U''_h\| \le \frac{\pi^2\mu^2}{h^2}\,.$$

2) The estimate for μ^2 follows from the fact that $a_i = \binom{2k}{2k+i}/\binom{2k}{k} < 1$, and that $\sum_{i=1}^\infty \frac{1}{i^2} = \frac{\pi^2}{6}$:

$$\frac{\pi^2}{8}\mu^2 = \sum_{\text{odd}\, i}^k \frac{a_i}{i^2} < \sum_{\text{odd}\, i}^\infty \frac{1}{i^2} = \sum_{i=1}^\infty \frac{1}{i^2} - \sum_{\text{even}\, i}^\infty \frac{1}{i^2} = \Big(1-\frac{1}{4}\Big)\sum_{i=1}^\infty \frac{1}{i^2} = \frac{\pi^2}{8}\,.$$

We will prove in §6 that $1-\mu_{2k}^2 \asymp \frac{1}{\sqrt{2k}}$. □

Lemma 4.2 *We have*

$$\|U_h^m\|_{T_{n-1}^\perp} \le F_{2m}\Big(\frac{\pi^2\mu^2}{n^2h^2}\Big)^m\,. \tag{4.4}$$

Proof. 1) If f is orthogonal to T_{n-1}, then so are its Steklov functions $I_{ih}(f)$, hence $U_h(f)$ and the iterates $U_h^m(f)$ as well. Also, the operators D^2 (of double differentiation) and U_h commute (since D^2 and I_{ih} clearly commute). Therefore, using the Bohr-Favard inequality with the $(2m)$-th derivative, we obtain

$$\|U_h^m(f)\|_{T_{n-1}^\perp} \le \frac{F_{2m}}{n^{2m}}\,\|D^{2m}U_h^m(f)\| = \frac{F_{2m}}{n^{2m}}\,\|\,[D^2U_h]^m(f)\| \le \frac{F_{2m}}{n^{2m}}\,\|D^2U_h\|^m\|f\|\,. \tag{4.5}$$

By (4.2), we get $\|D^2U_h\| \le \frac{\pi^2\mu^2}{h^2}$, hence the conclusion. □

Remark 4.3 For $h = \frac{\pi}{n}$, we have equality in (4.4), i.e.,

$$\|U_h^m\|_{T_{n-1}^\perp} = F_{2m}\mu^{2m}, \qquad h = \frac{\pi}{n}\,,$$

which is attained on the Favard function $\varphi_n(x) = \operatorname{sgn}\sin nx$. Indeed, for $h = \frac{\pi}{n}$, we have

$$\varphi_n(x-ih) - 2\varphi_n(x) + \varphi_n(x+ih) = \begin{cases} -4\varphi_n(x), & \text{odd } i, \\ 0, & \text{even } i, \end{cases}$$

and it follows from (4.3) that $U''_h(\varphi_n) = -\frac{\pi^2\mu^2}{h^2}\varphi_n$, and respectively

$$D^{2m}U_h^m(\varphi_n) = (-1)^m\Big(\frac{\pi^2\mu^2}{h^2}\Big)^m\varphi_n\,.$$

On the other hand, the Bohr-Favard inequality turns into equality on the functions $f \in T_{n-1}^\perp$ such that $f^{(2m)}(x) = a\varphi_n(x-b)$, hence on $U_h^m(\varphi_n)$. Therefore, in (4.5), we have equalities all the way through.

Proposition 4.4 *Let $f \in T_{n-1}^{\perp}$, and let $h > \frac{\pi}{n}\mu$. Then*

$$\|f\| \le c_{n,2k}(h)\, \gamma_{2k}^{*}\, \omega_{2k}(f,h)\,, \tag{4.6}$$

where

$$c_{n,2k}(h) = \Big(\cos\frac{\pi}{2}\rho\Big)^{-1}, \qquad \rho = \frac{\pi\mu}{nh} < 1. \tag{4.7}$$

Proof. From Proposition 3.2, using the estimate (4.4), we obtain

$$c_{n,2k}(h) = \sum_{m=0}^{\infty} \|U_h^m\|_{T_{n-1}^{\perp}} \le \sum_{m=0}^{\infty} F_{2m}\rho^{2m} = \Big(\cos\frac{\pi}{2}\rho\Big)^{-1},$$

the last equality (provided $\rho < 1$) being the Taylor expansion of $\sec\frac{\pi}{2}x = 1/\cos\frac{\pi}{2}x$. (The latter is usually given in terms of the Euler numbers E_{2m} as $\sec x = \sum_{m=0}^{\infty}\frac{|E_{2m}|}{(2m)!}x^{2m}$, see, e.g., Gradshteyn, Ryzhik [6, § 1. 411. 9], so we have $\sec\frac{\pi}{2}x = \sum_{m=0}^{\infty}\frac{|E_{2m}|\pi^{2m}}{2^{2m}(2m)!}x^{2m}$, and we use the fact that $F_{2m} = \frac{|E_{2m}|\pi^{2m}}{2^{2m}(2m)!}$, see [6, § 0. 233. 6].) □

Theorem 4.5 *If $f \in T_{n-1}^{\perp}$, then, for any $\alpha > 1$, we have*

$$\|f\| \le c_{\alpha}\, \gamma_{2k}^{*}\, \omega_{2k}(f, \tfrac{\alpha\pi}{n})\,, \qquad c_{\alpha} = \Big(\cos\frac{\pi}{2\alpha}\Big)^{-1}. \tag{4.8}$$

Proof. Just put $h = \frac{\alpha\pi}{n}$ in (4.6), and use the fact that $\mu < 1$. □

Let us give some particular cases of Theorem 4.5.

$$\begin{array}{llll}
1) & \alpha = 2, & c_{\alpha} = (\cos\frac{\pi}{4})^{-1} = \sqrt{2}, & \|f\| \le 1\frac{1}{2}\, \gamma_{2k}^{*}\, \omega_{2k}(f, \frac{2\pi}{n})\,; \\
2) & \alpha = \frac{3}{2}, & c_{\alpha} = (\cos\frac{\pi}{3})^{-1} = 2, & \|f\| \le 2\, \gamma_{2k}^{*}\, \omega_{2k}(f, \frac{3\pi}{2n})\,; \\
3) & \alpha = \frac{4}{3}, & c_{\alpha} = (\cos\frac{3\pi}{8})^{-1} = 2.61, & \|f\| \le 2\frac{2}{3}\, \gamma_{2k}^{*}\, \omega_{2k}(f, \frac{4\pi}{3n})\,; \\
4) & \alpha = \frac{5}{4}, & c_{\alpha} = (\cos\frac{2\pi}{5})^{-1} = 3.23, & \|f\| \le 3\frac{1}{4}\, \gamma_{2k}^{*}\, \omega_{2k}(f, \frac{5\pi}{4n})\,.
\end{array} \tag{4.9}$$

From the relations $\cos\frac{\pi}{2}x = \sin\frac{\pi}{2}(1-x) \ge \frac{\pi}{4}(1-x^2)$, it follows that, in (4.8),

$$c_{\alpha} < \frac{4}{\pi}\Big(1 - \frac{1}{\alpha^2}\Big)^{-1},$$

i.e., c_{α} behaves like $\frac{2}{\pi}\frac{1}{\alpha-1}$ as $\alpha \searrow 1$.

Theorem 4.6 *If $f \in T_{n-1}^{\perp}$, then, for $\delta = \frac{\pi}{n}$, we have*

$$\|f\| \le c_{2k}\, \gamma_{2k}^{*}\, \omega_{2k}(f, \tfrac{\pi}{n})\,, \qquad c_{2k} = \mathcal{O}(\sqrt{2k})\,. \tag{4.10}$$

Proof. Putting $h = \frac{\pi}{n}$ into (4.6), we obtain the inequality (4.10) with the constant

$$c_{2k} = \Big(\cos\frac{\pi}{2}\mu_{2k}\Big)^{-1} < \frac{4}{\pi}\Big(1 - \mu_{2k}^2\Big)^{-1}, \tag{4.11}$$

and we are proving in §6 that $1 - \mu_{2k}^2 \asymp \frac{1}{\sqrt{2k}}$. □

5 Stechkin inequality for $\frac{\pi}{n} < \delta \le \frac{2\pi}{n}$

1) Consider the de la Vallée Poussin sum (operator)

$$v_{m,n} = \frac{1}{n-m} \sum_{i=m}^{n-1} s_i \,, \tag{5.1}$$

which is an average of $(n-m)$ Fourier sums s_i of degree i. For $m = n-1$ and for $m = 0$, it becomes the Fourier sum s_{n-1} and the Fejer sum $\sigma_n = \frac{1}{n}\sum_{i=0}^{n-1} s_i$, respectively.

Since $v_{m,n}(f)$ is the convolution of f with the de la Vallée Poussin kernel $V_{m,n}$, we clearly have

$$\omega_k(v_{m,n}(f), \delta) \le \|v_{m,n}\|\, \omega_k(f, \delta),$$

where $\|v_{m,n}\|$ is the norm, or the Lebesgue constant, of the operator $v_{m,n}$.

Stechkin [11] made a detailed studies of behaviour of the value $\|v_{m,n}\|$ as a function of m and n. We will need just two facts from his work, one of them combined with a later result of Galkin [5].

a) The norm $\|v_{m,n}\|$ depends only on ratio m/n, and in a monotone way. Precisely, with

$$\ell(x) := \frac{2}{\pi} \int_0^\infty \frac{|\sin xt \cdot \sin t|}{t^2}\, dt,$$

which is (non-trivially) a monotonely increasing function of x, we have

$$\|v_{m,n}\| = \ell(x_{m/n}), \qquad x_{m/n} := \frac{1+m/n}{1-m/n}\,.$$

b) The values of ℓ at integer points can be related to the so-called Watson constants $L_{M/2}$ (for $M = 2N$, they turn into the Lebesgue constants $L_N := \|s_N\|$ of the Fourier operator s_N). Namely,

$$\ell(M+1) = L_{M/2}\,,$$

and from the result of Galkin [5] that $L_{M/2} < \frac{4}{\pi^2}\ln(M+1) + 1$, we conclude that

$$\ell(p) < \frac{4}{\pi^2} \ln p + 1 \qquad \text{for integer } p, \tag{5.2}$$

therefore (rather roughly)

$$\ell(x) < \frac{4}{\pi^2} \ln(x+1) + 1 \qquad \text{for all } x. \tag{5.3}$$

2) Now, from definition (5.1), we see firstly that $v_{m,n}(f)$ is a trigonometric polynomial of degree $\le n-1$, hence

$$E_{n-1}(f) \le \|f - v_{m,n}(f)\|,$$

and secondly that $v_{m,n}$ acts as identity on T_m, therefore

$$f - v_{m,n}(f) \perp T_m\,.$$

So, we may apply Proposition 4.4 to the difference $f - v_{m,n}(f)$ to obtain

$$\begin{aligned} E_{n-1}(f) &\le \|f - v_{m,n}(f)\| \\ &\le c_{m+1,2k}(h)\, \gamma_{2k}^*\, \omega_{2k}\Big(f - v_{m,n}(f), h\Big) \\ &\le c_{m+1,2k}(h)\, (1 + \|v_{m,n}\|)\, \gamma_{2k}^*\, \omega_{2k}(f, h) \\ &= \Big[\cos\Big(\frac{\pi}{2}\frac{\pi\mu}{(m+1)h}\Big)\Big]^{-1} \Big[1 + \ell\Big(\frac{1+m/n}{1-m/n}\Big)\Big]\, \gamma_{2k}^*\, \omega_{2k}(f, h)\,. \end{aligned}$$

Now, with some parameter $s \in [0, 1)$ which may well depend on n and h, we put in the last line

$$m = \lfloor sn \rfloor.$$

With such an m, we have $m + 1 > sn$ and $m/n \le s$, therefore

$$E_{n-1}(f) \le \Big[\cos\Big(\frac{\pi}{2}\frac{\mu}{s}\frac{\pi}{nh}\Big)\Big]^{-1} \Big[1 + \ell\Big(\frac{1+s}{1-s}\Big)\Big] \gamma_{2k}^*\, \omega_{2k}(f, h)\,. \tag{5.4}$$

Finally, taking $h = \frac{\alpha\pi}{n}$, and evaluating the factor $1 + \ell(x_s)$ with the help of (5.3), we obtain

$$E_{n-1}(f) \le \Big(\cos\frac{\pi\mu}{2\alpha s}\Big)^{-1} \Big[2 + \frac{4}{\pi^2}\ln\Big(\frac{2}{1-s}\Big)\Big] \gamma_{2k}^*\, \omega_{2k}\Big(f, \frac{\alpha\pi}{n}\Big)\,, \tag{5.5}$$

where we can minimize the right-hand side with respect to $s \in (\frac{\mu}{\alpha}, 1)$.

3) Now, using the last estimate, we establish Stechkin inequalities for particular α's.

Theorem 5.1 *For all $n \ge 1$, we have*

$$E_{n-1}(f) \le c\, \gamma_{2k}^*\, \omega_{2k}\Big(f, \frac{2\pi}{n}\Big)\,, \qquad c = 5\,.$$

Proof. In (5.5), take $\alpha = 2$ and majorize μ by 1. Then the constant for $\delta = \frac{2\pi}{n}$ takes the form

$$c = \Big(\cos\frac{\pi}{4s}\Big)^{-1} \Big[2 + \frac{4}{\pi^2}\ln\Big(\frac{2}{1-s}\Big)\Big]\,.$$

It turns out that the value $s = 8/9$ is almost optimal, and we obtain Stechkin inequality with the constant

$$c = \Big(\cos\frac{9\pi}{32}\Big)^{-1} \Big[2 + \frac{4}{\pi^2}\ln 18\Big] = 4.999144 < 5. \tag{5.6}$$

To make sure that our step away from 5 is free from a round-off error, we notice that, for $s = \frac{8}{9}$, we have in (5.4)

$$\ell\Big(\frac{1+s}{1-s}\Big) = \ell(17) = L_8\,.$$

Therefore, in the pass from (5.4) to (5.5), we can use the estimate (5.2) instead of (5.3), thus changing in (5.6) the value $\ln 18$ to $\ln 17$, and that will give the constant $c = 4.962628$. We can make another bit down by computing directly the Lebesgue constant $L_8 = 2.137730$, hence getting

$$c = \Big(\cos\frac{9\pi}{32}\Big)^{-1} \Big[1 + L_8\Big] = 4.946034,$$

so that $c < 5$ is secured. $\square$

Remark 5.2 Surprising is the fact that, in this theorem, the upper estimate is provided by one and the same linear method of approximation that works for all r simultaneously. Namely, for any r, the de la Vallée Poussin operator $v_{m,n}$ with $m = \lfloor \frac{8}{9} n \rfloor$ provides

$$\|f - v_{m,n}(f)\| \le 5\, \gamma_r^* \omega_r\Big(f, \frac{2\pi}{n}\Big), \qquad \forall r \in \mathbb{N}.$$

Perhaps it makes sense to try to derive such an estimate directly from the properties of $v_{m,n}$.

Theorem 5.3 *For any $\alpha > 1$, there exists a constant c_α that depends only on α such that*

$$E_{n-1}(f) \le c_\alpha\, \gamma_{2k}^*\, \omega_{2k}\Big(f, \frac{\alpha\pi}{n}\Big), \qquad n \ge 1. \tag{5.7}$$

Proof. Putting (a non-optimal) $s = \frac{1}{\sqrt{\alpha}}$ in (5.5), and again majorizing μ by 1, we obtain (5.7) with

$$\begin{aligned} c_\alpha &= \Big(\cos\frac{\pi}{2\sqrt{\alpha}}\Big)^{-1} \Big(\frac{4}{\pi^2} \ln\Big(\frac{2\sqrt{\alpha}}{\sqrt{\alpha}-1}\Big) + 2\Big) \\ &\le \frac{4}{\pi}\frac{\alpha}{\alpha-1} \Big(\frac{4}{\pi^2} \ln\Big(\frac{2\sqrt{\alpha}}{\sqrt{\alpha}-1}\Big) + 2\Big), \end{aligned}$$

where we have used the inequality $\cos\frac{\pi}{2}x \ge \frac{\pi}{4}(1 - x^2)$ for $|x| \le 1$. □

6 Stechkin inequality for $\delta = \frac{\pi}{n}$

Theorem 6.1 *For $\delta = \frac{\pi}{n}$, and $r = 2k$, we have*

$$E_{n-1}(f) \le c_r(\tfrac{\pi}{n})\, \gamma_r^*\, \omega_r\Big(f, \frac{\pi}{n}\Big), \qquad n \ge 1, \tag{6.1}$$

where

$$c_r(\tfrac{\pi}{n}) = \mathcal{O}(\sqrt{r}\ln r)\,. \tag{6.2}$$

Proof. From the estimate (5.5), with $h = \frac{\pi}{n}$ and $s = \sqrt{\mu}$, we obtain the inequality (6.1) with the constant

$$\begin{aligned} c_{2k}(\tfrac{\pi}{n}) &= \Big(\cos\frac{\pi}{2}\sqrt{\mu}\Big)^{-1} \Big(\frac{4}{\pi^2} \ln\Big(\frac{2}{1-\sqrt{\mu}}\Big) + 2\Big) \\ &< \frac{4}{\pi}\frac{1}{1-\mu} \Big(\frac{4}{\pi^2} \ln\Big(\frac{2}{1-\sqrt{\mu}}\Big) + 2\Big). \end{aligned}$$

The estimate (6.2) follows now from the fact that

$$1 - \mu_{2k}^2 > \frac{c_1}{\sqrt{2k}}\,, \qquad c_1 = \frac{2}{3},$$

which we are proving in the next lemma. With the value $c_1 = \frac{2}{3}$ at hands, we can give the explicit estimate $c_r(\frac{\pi}{n}) < 2\sqrt{r}\ln r + 12\sqrt{r}$. □

Lemma 6.2 *For* $\mu_{2k}^2 := \frac{8}{\pi^2}\sum_{\mathrm{odd}\, i}^{k} \frac{a_i}{i^2}$, *where* $a_i := \binom{2k}{k+i}/\binom{2k}{k}$, *we have*

$$\frac{c_1}{\sqrt{2k}} < 1 - \mu_{2k}^2 < \frac{c_2}{\sqrt{2k}}\,, \qquad c_1 = \frac{2}{3}, \quad c_2 = \frac{5}{4}\,. \tag{6.3}$$

Proof. Let us compute the value $\widehat{\Delta}_t^{2k}(f_0, x)$ for $f_0(x) = \cos x$ at $x = 0$. Since

$$\widehat{\Delta}_t^2(\cos, x) = -\cos(x-t) + 2\cos x - \cos(x+t) = 2\cos x(1-\cos t) = 4\sin^2\tfrac{t}{2}\cos x\,,$$

we have

$$\widehat{\Delta}_t^{2k}(f_0, x)\Big|_{x=0} = 4^k \sin^{2k}\tfrac{t}{2}\,.$$

On the other hand, by the definition,

$$\widehat{\Delta}_t^{2k}(f_0, x)\Big|_{x=0} = \sum_{i=-k}^{k} (-1)^i \binom{2k}{k+i}\cos(x+it)\Big|_{x=0} = \binom{2k}{k}\Big[1 - 2\sum_{i=1}^{k}(-1)^{i+1}a_i\cos it\Big].$$

So, we have

$$1 - 2\sum_{i=1}^{k}(-1)^{i+1}a_i\cos it = \lambda_k\sin^{2k}\tfrac{t}{2}, \qquad \lambda_k := \frac{4^k}{\binom{2k}{k}}\,.$$

Integrating both parts twice, first time between 0 and u, and then between 0 and π, we obtain: for the left-hand side

$$\Big[\frac{u^2}{2} + 2\sum_{i=1}^{k}(-1)^{i+1}\frac{a_i}{i^2}\cos iu\Big]_0^{\pi} = \frac{\pi^2}{2} - 4\sum_{\mathrm{odd}\, i}^{k}\frac{a_i}{i^2} = \frac{\pi^2}{2}(1-\mu_{2k}^2)\,,$$

and for the right-hand side

$$\lambda_k\int_0^{\pi}\int_0^{u}\sin^{2k}(\tfrac{t}{2})\,dt\,du = \lambda_k\int_0^{\pi}(\pi - t)\sin^{2k}(\tfrac{t}{2})\,dt = 4\lambda_k\int_0^{\pi/2}\tau\cos^{2k}(\tau)\,d\tau$$

(we firstly changed the order of integration and then put $\tau = \frac{\pi}{2} - \frac{t}{2}$). So, equating the rightmost values in the last two lines, we obtain

$$1 - \mu_{2k}^2 = \frac{8}{\pi^2}\frac{4^k}{\binom{2k}{k}}\int_0^{\pi/2} t\cos^{2k}t\,dt\,. \tag{6.4}$$

Now, by Wallis inequality, we have

$$\sqrt{\frac{\pi}{2}}\sqrt{2k} \le \frac{4^k}{\binom{2k}{k}} \le \sqrt{\frac{\pi}{2}}\sqrt{2k+1}\,,$$

while the integral admits the two-sided estimate

$$\frac{1}{2k+1} \le \int_0^{\pi/2} t\cos^{2k}(t)\,dt \le \frac{1}{2k}\,,$$

because $\sin t \le t \le \frac{\sin t}{\cos t}$ on $[0, \frac{\pi}{2}]$, and $\int_0^{\pi/2}\sin(t)\cos^m(t)\,dt = \frac{1}{m+1}$. Hence

$$\frac{8}{\pi^2}\sqrt{\frac{\pi}{2}}\frac{\sqrt{2k}}{2k+1} \le 1 - \mu_{2k}^2 \le \frac{8}{\pi^2}\sqrt{\frac{\pi}{2}}\frac{\sqrt{2k+1}}{2k}\,,$$

and (6.3) follows with $c_1 = \frac{8}{\pi^2}\sqrt{\frac{\pi}{2}}\frac{2k}{2k+1} > \frac{2}{3}$ and $c_2 = \frac{8}{\pi^2}\sqrt{\frac{\pi}{2}}\sqrt{\frac{2k+1}{2k}} < \frac{5}{4}$. □

7 On the factor $\sqrt{r}$ at $\delta = \frac{\pi}{n}$

For $\delta = \frac{\pi}{n}$, our estimates for the Stechkin constant (with the lower bound yet to be proved) look as follows:

$$c'\gamma_r^* \le K_{n,r}(\tfrac{\pi}{n}) \le c\sqrt{r}\ln r\, \gamma_r^*\,,$$

i.e., the upper and lower bounds do not match. In §2 we already expressed our belief that additional factors on the right are redundant. However, as we show in this section, appearance of the factor $\sqrt{r}$ within our method is unavoidable. (The factor $\ln r$ originates from the use of the de la Vallée Poussin sums, and perhaps can be removed by some more sophisticated technique.)

From our initial steps (3.2)-(3.4), it is easy to see that our upper estimates in all Stechkin inequalities are valid not only for the standard modulus of smoothness $\omega_{2k}(f,h)$, but also for the modulus

$$\omega_{2k}^*(f,h) := \Big\| \int_{\mathbb{R}} \widehat{\Delta}_t^{2k}(f,\cdot)\phi_h(t)\,dt \Big\|\,, \tag{7.1}$$

which has a smaller value at every h. It is clear that the Stechkin constant defined with respect to a smaller modulus takes larger values, and now we show that, for the modulus $\omega_{2k}^*(f,h)$, the increase at $h = \frac{\pi}{n}$ is exactly by the factor $\sqrt{2k}$.

Theorem 7.1 *For $r = 2k$, we have*

$$\frac{\gamma_r^*}{1-\mu_r^2} \le \sup_{f\in T_{n-1}^\perp} \frac{\|f\|}{\omega_r^*(f,\frac{\pi}{n})} \le \frac{4}{\pi}\,\frac{\gamma_r^*}{1-\mu_r^2}\,,$$

where

$$\frac{\gamma_r^*}{1-\mu_r^2} \asymp \sqrt{r}\,\gamma_r^* \asymp \frac{r}{2^r}\,.$$

Proof. The upper bound was established in (4.10)-(4.11). For the lower bound, take $f_0(x) = \cos nx$. Then

$$\widehat{\Delta}_t^{2k}(f_0,x) = 4^k \sin^{2k}\left(\frac{nt}{2}\right)\cos nx, \qquad \phi_{\pi/n}(t) = \frac{n}{\pi}\Big(1-\frac{n}{\pi}|t|\Big), \qquad |t| \le \frac{\pi}{n}\,, \tag{7.2}$$

hence

$$\begin{aligned}
\omega_{2k}^*(f_0,\tfrac{\pi}{n}) &= \Big\| \int_{-\pi/n}^{\pi/n} \Delta_t^{2k}(f_0,\cdot)\,\phi_{\pi/n}(t)\,dt \Big\| \\
&= 2\cdot 4^k \int_0^{\pi/n} \sin^{2k}\left(\frac{nt}{2}\right)\frac{n}{\pi}\Big(1-\frac{n}{\pi}\,t\Big)dt \\
&= \frac{8}{\pi^2}\,4^k \int_0^{\pi/2} \tau\cos^{2k}(\tau)\,d\tau \qquad \Big(\tau = \frac{\pi}{2}-\frac{nt}{2}\Big) \\
&\overset{(6.4)}{=} \frac{1-\mu_{2k}^2}{\gamma_{2k}^*}\,,
\end{aligned}$$

while $\|f_0\| = 1$. $\square$

Since also $E_{n-1}(f_0) = 1$, we have the same estimate for the ratio $E_{n-1}(f_0)/\omega_{2k}^*(f_0, \frac{\pi}{n})$, therefore, for the Stechkin constant $K_{n,r}^*(\delta)$ defined with respect to the modulus $\omega_{2k}^*(f, \delta)$, we obtain at $\delta = \frac{\pi}{n}$

$$c' \sqrt{r}\, \gamma_r^* \le K_{n,r}^*(\tfrac{\pi}{n}) := \sup_{f \in C} \frac{E_{n-1}(f)}{\omega_r^*(f, \frac{\pi}{n})} \le c \sqrt{r} \ln r\, \gamma_r^* \,.$$

8 Lower estimate

Lemma 8.1 *For any n, r and ϵ, and for any $\delta < \frac{\pi}{r}$, there exists an $f \in C$ such that,*

$$E_{n-1}(f) \ge \frac{1}{2} \gamma_{r-1}^* \, \omega_r(f, \delta) - \epsilon \,.$$

Proof. Take the step periodic function

$$f_0(x) = \begin{cases} 1, & x \in (-\pi, 0]; \\ 0, & x \in (0, \pi]. \end{cases}$$

For any $x \in [-\pi, \pi]$, and for any $h < \frac{\pi}{r}$, consider the values of this function at the points $x_i = x + ih$, where $0 \le i \le r$. It is clear that, for some $m \le r$, we have either

$$f_0(x_i) = 1, \quad 0 \le i \le m, \qquad f_0(x_i) = 0, \quad m < i \le r,$$

or the other way round. Therefore, for the modulus of smoothness $\omega_r(f_0, \delta)$, we have the following relations:

$$\begin{aligned} \omega_r(f_0, \delta) &= \max_{0<h\le\delta} \max_x |\Delta_h^r f_0(x)| = \max_{0<h\le\delta} \max_x \Big| \sum_{i=0}^r (-1)^i \binom{r}{i} f_0(x+ih) \Big| \\ &= \max_{0\le m\le r} \Big| \sum_{i=0}^m (-1)^i \binom{r}{i} \Big| = \max_{0\le m\le r} \Big| (-1)^m \binom{r-1}{m} \Big| = \binom{r-1}{\lfloor \frac{r-1}{2} \rfloor} = 1/\gamma_{r-1}^* \,, \end{aligned}$$

i.e.,

$$\omega_r(f_0, \delta) = 1/\gamma_{r-1}^* \,.$$

It is also clear that, for the best L_∞-approximation of f_0, we have

$$E_{n-1}(f_0) = \frac{1}{2},$$

therefore the result for such an f_0 (without ϵ subtracted).

This is almost what we need except that f_0 is not continuous. But we can get a continuous f by smoothing f_0 at the points of discontinuity, say, by linearization. For a given ϵ, set

$$f(x) = \frac{1}{\epsilon} \int_{-\epsilon/2}^{\epsilon/2} f_0(x+t)\, dt.$$

i.e.,

$$f(x) = \begin{cases} 1, & x \in [-\pi+\epsilon, -\epsilon]; \\ 0, & x \in [\epsilon, \pi-\epsilon]; \\ \text{is linear on } [-\epsilon,\epsilon] \text{ and } [\pi-\epsilon, \pi+\epsilon]. \end{cases}$$

Then, from the definition (or, more generally, because f is the convolution of f_0 with a positive kernel), it folows that

$$\omega_r(f,\delta) \le \omega_r(f_0,\delta) = 1/\gamma^*_{r-1}\,.$$

As for the best approximation of f, we have

$$E_{n-1}(f) \ge \frac{1}{2} - \epsilon'\,.$$

Indeed, since $E_{n-1}(f) = \|f - t_{n-1}\| \le \|f\| = 1$, the polynomial t_{n-1} of best approximation satisfies $\|t_{n-1}\| \le 2$, therefore, by Bernstein inequality, we have $\|t'_{n-1}\| \le 2(n-1)$, hence, on the interval $[-\epsilon,\epsilon]$ of the length 2ϵ the range of t_{n-1} is not more than $4(n-1)\epsilon =: 2\epsilon'$, while the function f on the same interval takes the values 0 and 1. □

Theorem 8.2 *For any r, and any $\delta \le \frac{\pi}{r}$, we have*

$$K_{n,r}(\delta) := \sup_{f\in C} \frac{E_{n-1}(f)}{\omega_r(f,\delta)} \ge c'_r\,\gamma^*_r$$

where

$$c'_r = \begin{cases} \frac{r}{r+1}, & r = 2k-1; \\ 1, & r = 2k. \end{cases}$$

In particular, for any r and any $n \ge 2r$ (i.e., when $\frac{2\pi}{n} \le \frac{\pi}{r}$),

$$K_{n,r}(\tfrac{2\pi}{n}) := \sup_{f\in C} \frac{E_{n-1}(f)}{\omega_r(f,\frac{2\pi}{n})} \ge c'_r\,\gamma^*_r, \qquad n \ge 2r.$$

Proof. The first lower bound is just a reformulation of the previous lemma, because, for $\gamma^*_r := \binom{r}{\lfloor \frac{r}{2}\rfloor}^{-1}$, we have $\frac{1}{2}\,\gamma^*_{r-1} = c'_r\,\gamma^*_r\,.$ □

Remark 8.3 The order $r^{1/2}2^{-r}$ of the lower bound for the Stechkin constant was established earlier by Ivanov [7], but he did not pay attention to the constant (and his extremal function was different from ours).

9 Stechkin constants for small r

For small $r = 2k$, when μ_r is noticeably smaller than 1, our method in §5 will give for the Stechkin constant the upper estimates which are better than $5\gamma^*_r$, but they will never be smaller than $2\gamma^*_r$ because of the factor $1 + \|v_{m,n}\|$.

Surprisingly, better values (for small r) which stand quite close to the lower bound $1 \cdot \gamma_r^*$ could be obtained through technique of intermediate approximation with Steklov-type functions. (For general r, this technique provides the same overblown estimate $c_r < r^{ar}$ as Stechkin's original proof, therefore a surprise.)

Such a technique is of course well-known (it was introduced probably by Brudnyi [1] and Freud–Popov [4]), and it was exploited repeatedly for proving Stechkin inequalities of various types (e.g., for spline and one-sided approximations). Our only innovation (if any) is the use of the central differences instead of the forward ones, which reduces the constants by the factor $\binom{2k}{k}$, and the will to take a closer look at their actual values.

Lemma 9.1 *We have*

$$E_{n-1}(f) \le c_{2k}\Big(\frac{\alpha\pi}{n}\Big)\,\gamma_{2k}^*\,\omega_{2k}\Big(f, \frac{\alpha\pi}{n}\Big)\,,$$

where

$$c_{2k}\Big(\frac{\alpha\pi}{n}\Big) = 1 + F_{2k}\frac{k^{2k}}{(\alpha\pi)^{2k}}\sum_{i=1}^{k}\frac{2b_i}{i^{2k}}, \qquad b_i = \binom{2k}{k+i}\,, \tag{9.1}$$

and F_{2k} are the Favard constants.

Proof. Given f, with any $2k$ times differentiable function f_h, we have

$$E_{n-1}(f) \;\le\; E_{n-1}(f - f_h) + E_{n-1}(f_h) \;\le\; \|f - f_h\| + \frac{F_{2k}}{n^{2k}}\,\|f_h^{(2k)}\| \tag{9.2}$$

where we used the Favard inequality for the best approximations of f_h. A typical choice of f_h is via the Steklov functions of order $2k$:

$$I_{ih}(f,x) := \frac{1}{(h/k)^{2k}}\underbrace{\int_{-h/2k}^{h/2k}\cdots\int_{-h/2k}^{h/2k}}_{2k} f(x - i(t_1+\cdots+t_{2k}))\,dt_1\cdots dt_{2k}\,,$$

$$I_{ih}^{(2k)}(f,x) = \frac{(-1)^k}{(ih/k)^{2k}}\widehat{\Delta}_{ih/k}^{2k}f(x)\,,$$

namely

$$f_h := \frac{1}{\binom{2k}{k}}\sum_{\substack{i=-k\\ i\ne 0}}^{k}(-1)^{i+1}\binom{2k}{k+i}I_{ih}(f) = \gamma_{2k}^*\sum_{i=1}^{k}(-1)^{i+1}2b_iI_{ih}(f)\,.$$

Then

$$\|f - f_h\| \le \gamma_{2k}^*\,\omega_{2k}(f,h),$$
$$\|f_h^{(2k)}\| \le \gamma_{2k}^*\sum_{i=1}^{k}\frac{2b_i}{(ih/k)^{2k}}\,\omega_{2k}(f, ih/k) \;\le\; \gamma_{2k}^*\,\omega_{2k}(f,h)\,\frac{k^{2k}}{h^{2k}}\sum_{i=1}^{k}\frac{2b_i}{i^{2k}}\,,$$

whence applying (9.2)

$$E_{n-1}(f) \;\le\; c_{2k}(h)\,\gamma_{2k}^*\,\omega_2(f,h)\,, \qquad c_{2k}(h) = 1 + F_{2k}\frac{k^{2k}}{(nh)^{2k}}\sum_{i=1}^{k}\frac{2b_i}{i^{2k}}\,,$$

and we take $h = \frac{\alpha\pi}{n}$. $\square$

In (9.1), we can obtain a small value only if $\frac{k}{\alpha\pi} < 1$, i.e., we may try $k = (1,2,3)$ for $\alpha = 1$, and $k = (1,2,3,4,5)$ for $\alpha = 2$. So we did (dropping those values for which the resulting constants in (9.1) were not close to 1).

Theorem 9.2 *For $\delta = \frac{\pi}{n}$ and $\delta = \frac{2\pi}{n}$, we have*

$$E_{n-1}(f) \le c_r(\delta)\,\gamma_r^*\,\omega_r(f,\delta)\,,$$

where $c_{2k-1}(\delta) = c_{2k}(\delta)$, and the values of $c_{2k}(\delta)$ are given below

$c_2(\frac{\pi}{n})$	$c_4(\frac{\pi}{n})$
$1\frac{1}{4}$	$2\frac{7}{10}$

,

$c_2(\frac{2\pi}{n})$	$c_4(\frac{2\pi}{n})$	$c_6(\frac{2\pi}{n})$
$1\frac{1}{16}$	$1\frac{1}{9}$	$1\frac{1}{2}$

.

Proof. We will use the following values: $F_2 = \frac{\pi^2}{8}$, $F_4 = \frac{5\pi^4}{384}$, $F_6 = \frac{61\pi^6}{46080}$.

1) For $2k = 2$, we have

$$c_2\Big(\frac{\alpha\pi}{n}\Big) = 1 + \frac{\pi^2}{8}\frac{2}{(\alpha\pi)^2} = 1 + \frac{1}{4\alpha^2}\,.$$

With $\alpha = 1$ and $\alpha = 2$, we obtain $c_2(\frac{\pi}{n}) = \frac{5}{4}$ and $c_2(\frac{2\pi}{n}) = \frac{17}{16}$. Also, with $\alpha = \frac{1}{2}$, we obtain the remarkable inequality

$$E_{n-1}(f) \le 1\cdot\omega_2\Big(f, \frac{\pi}{2n}\Big)\,.$$

2) For $2k = 4$,

$$c_4\Big(\frac{\alpha\pi}{n}\Big) = 1 + \frac{5\pi^4}{384}\frac{2^4}{(\alpha\pi)^4}\cdot 2\left[\frac{4}{1^4} + \frac{1}{2^4}\right] = 1 + \frac{325}{192}\frac{1}{\alpha^4}\,.$$

With $\alpha = 1$ and $\alpha = 2$, we obtain $c_4(\frac{\pi}{n}) = \frac{517}{192} = 2.6927$, and $c_4(\frac{2\pi}{n}) = \frac{3397}{3072} = 1.1058$.

3) For $2k = 6$, with $\alpha = 2$, we have

$$c_6\Big(\frac{2\pi}{n}\Big) = 1 + \frac{61\pi^6}{46080}\frac{3^6}{(2\pi)^6}\cdot 2\left[\frac{15}{1^4} + \frac{6}{2^6} + \frac{1}{3^6}\right] = 1.4552 < 1\frac{1}{2}\,. \qquad \square$$

Theorem 9.2 provides a certain support to our Conjecture 2.1, which says, in particular, that, for even $r = 2k$, and for $\delta \ge \frac{\pi}{n}$, the best constant in the Stechkin inequality has the value $K_{n,r}(\delta) = 1\cdot\gamma_r^*$.

Acknowledgements. Our thanks to Alexander Babenko for his comments on a draft of this paper.

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